WAVE PROPAGATION ON THE SURFACE OF A HEAVY LIQUID IN A BASIN WITH UNEVEN BOTTOM

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As is known, the propagation velocity of long waves is proportional to $\sqrt{h_1}$ (h_1 is the basin depth). Therefore, when a wave propagates in a basin with uneven bottom its velocity is lower above the underwater prominences than above the deeper parts of the basin, which leads to deformation of the wave, accompanied by concentration of the energy above the shallow portions of the basin. In this case the wave propagating along an underwater ridge will differ considerably from the wave propagating above the deep portions of the basin. This peculiarity in the propagation of waves on the surface of a liquid was noted by Lavrent'ev (1957).

The first studies taking into account the described influence of an underwater ridge were made in the acoustic approximation by Munk and Arthur [1], who used the method of geometrical optics.

Sun Ts'ao made an experimental study of the influence of an underwater ridge and carried out some calculations [2]. He demonstrated the quantitative and qualitative discrepancy between observations and the calculations of Munk and Arthur. Thus, Sun Ts'ao notes nearly stationary propagation of the wave above the ridge. This result is not contained in the acoustic theory.

In this paper we examine the wave propagation problem in a basin with cylindrical bottom. In the nonlinear long-wave theory approximation it is shown that stationary propagation of a solitary wave above an underwater ridge may be observed for some bottom form. Nonstationary wave propagation above the ridge is examined in the linear theory. It is shown that for certain conditions the wave decays considerably more slowly along the ridge than in other directions.

1. Problem formulation. The subject problem reduces to finding the free surface form $z_1 = \zeta_1(x_1, y_1, t_1)$ and the velocity potential $\Phi_1(x_1, y_1, z_1, t_1)$. In the region occupied by the fluid the potential satisfies the Laplace equation

$$\frac{\partial^2\Phi_1}{\partial x_1^2} + \frac{\partial^2\Phi_1}{\partial y_1^2} + \frac{\partial^2\Phi_1}{\partial z_1^2} = 0; \tag{1.1}$$

and the boundary conditions:

1) bottom impermeability $z_1 = -h_1(x_1, y_1)$

$$\frac{\partial h_1}{\partial x_1}\frac{\partial\Phi_1}{\partial x_1} + \frac{\partial h_1}{\partial y_1}\frac{\partial\Phi_1}{\partial y_1} + \frac{\partial\Phi_1}{\partial z_1} = 0; \tag{1.2}$$

2) free surface impermeability $z_1 = \zeta_1(x_1, y_1, t_1)$

$$\frac{\partial\zeta_1}{\partial t_1} + \frac{\partial\zeta_1}{\partial x_1}\frac{\partial\Phi_1}{\partial x_1} + \frac{\partial\zeta_1}{\partial y_1}\frac{\partial\Phi_1}{\partial y_1} = \frac{\partial\Phi_1}{\partial z_1}; \tag{1.3}$$

3) constant pressure at the free surface $z_1 = \zeta_1(x_1, y_1, t_1)$

$$\frac{\partial\Phi_1}{\partial t_1} + \frac{1}{2}\left(\left(\frac{\partial\Phi_1}{\partial x_1}\right)^2 + \left(\frac{\partial\Phi_1}{\partial y_1}\right)^2\left(\frac{\partial\Phi_1}{\partial z_1}\right)^2\right) + g\zeta_1 = 0. \tag{1.4}$$

Here g is the gravity acceleration.

In addition, the solution must satisfy certain initial conditions, which can be formulated in terms of the initial value of the potential and the initial form of the free surface [3].

The problem presents serious difficulties for solution in this formulation. However, the problem can be simplified considerably by certain additional assumptions on the nature of the solution.

2. **Long waves in a basin with uneven bottom (nonlinear theory).** If the wavelength l is large in comparison with the average depth h_0 and the characteristic length of the bottom irregularities is no less than l, then we can introduce the small parameter $\alpha = h_0^2/l^2$ into the problem by making the variable replacement

$$h_1 = h_0(1 - \varepsilon h),$$
$$x_1 = lx, \quad y_1 = ly, \quad z_1 = h_0 z,$$
$$t_1 = \frac{l}{\sqrt{gh_0}} t, \quad \zeta_1 = a\zeta, \quad \Phi_1 = \frac{al}{h_0}\sqrt{gh_0}\Phi. \tag{2.1}$$

Here the time and potential scales are obtained from (1.1)–(1.4) under the assumption that ζ and Φ and their derivatives with respect to x, y, and t are of order unity.

We further assume that the wave amplitude

$$a = \alpha h_0 \tag{2.2}$$

as is the case for the solitary wave. The potential is sought in the form of the series

$$\Phi = \Phi_0 + \Phi_1\alpha + \Phi_2\alpha^2 + \ldots. \tag{2.3}$$

To find the coefficients of this series we need only know the value of the potential at the free surface $\varphi(x, y, t) = \Phi(x, y, \alpha\zeta, t)$ and use the Laplace equation and the bottom impermeability condition. The standard calculation yields

$$\frac{\partial}{\partial z}\Phi(x, y, z, t)|_{z=\alpha\zeta} = \alpha(\varepsilon\nabla h\nabla\varphi - (1 - \varepsilon h + \alpha\zeta)\Delta\varphi) -$$
$$- {}^1/_3\alpha^2\Delta^2\varphi + O(\alpha^3 + \alpha^2\varepsilon). \quad \left(\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{2.4}$$

Further

$$\nabla\Phi|_{z=\alpha\zeta} = \nabla\varphi + O(\alpha^2), \qquad \frac{\partial\Phi}{\partial t}\bigg|_{z=\alpha\zeta} = \frac{\partial\varphi}{\partial t} + O(\alpha^2). \tag{2.5}$$

Substituting (2.4) and (2.5) into the conditions at the free surface, we obtain two equations for determining the form of the free surface $\zeta(x, y, t)$ and the value of the potential at the free surface $\varphi(x, y, t)$

$$\frac{\partial\zeta}{\partial t} + \nabla((1 - \varepsilon h + \alpha\zeta)\nabla\varphi) + {}^1/_3\alpha\Delta^2\varphi = O(\alpha^2 + \varepsilon\alpha), \tag{2.6}$$

$$\frac{\partial\varphi}{\partial t} + {}^1/_2\alpha(\nabla\varphi)^2 + \zeta = O(\alpha^2). \tag{2.7}$$

Equation (2.6) is the condition of free surface impermeability, and Eq. (2.7) is the constant-pressure condition.

3. **Steady propagation of a solitary wave above a cylindrical bottom.** The steady solution of (2.6), (2.7) depends on y and t in the combination y − ct (in this section the combination y − ct is designated simply as y). Let for definiteness c > 0, i.e., the wave propagates in the positive direction of the y-axis.

The basin bottom is assumed cylindrical:

$$z = -1 + \varepsilon h(x) \quad (\varepsilon \ll 1).$$

We seek a solution of the form

$$\zeta(x, y) = \zeta_0(y) + \varepsilon\zeta_1(x, y) + O(\varepsilon^2), \tag{3.1}$$
$$\varphi(x, y) = \varphi_0(y) + \varepsilon\varphi_1(x, y) + O(\varepsilon^2). \tag{3.2}$$

Substitution of (3.1), (3.2) into (2.6), (2.7) leads to the known Rayleigh-Lavrent'ev formulas [4, 5] for the functions ζ_0 and φ_0.

One of the solutions of these equations will be the solitary wave

$$\zeta_0 = \mathrm{sch}^2\frac{\sqrt{3}y}{2}, \quad \frac{d\varphi_0}{dy} = \mathrm{sch}^2\frac{\sqrt{3}y}{2} + O(\alpha). \tag{3.3}$$

The wave propagation velocity c is connected with the wave amplitude by the relation

$$c = 1 + {}^1/_2\alpha + O(\alpha^2). \tag{3.4}$$

The functions ζ_1 and φ_1 satisfy the equations

$$-\frac{\partial \zeta_1}{\partial y} + \Delta\varphi_1 - h(x)\frac{d^2\varphi_0}{dy^2} = O(\alpha), \qquad -\frac{\partial\varphi_1}{\partial y} + \zeta_1 = O(\alpha). \tag{3.5}$$

Hence it is not difficult to obtain the wave profile distortion caused by the bottom irregularity

$$\zeta_1 = \frac{d^3\varphi_0}{dy^3}\int_{-\infty}^{x}(x-\xi)h(\xi)\,d\xi + O(\alpha). \tag{3.6}$$

A stationary solution which becomes a solitary wave as $|x| \to \infty$ is possible only in a basin whose bottom profile satisfies the conditions

$$\int_{-\infty}^{\infty} h(x)\,dx = 0, \qquad \int_{-\infty}^{\infty} xh(x)\,dx = 0. \tag{3.7}$$

These conditions will be satisfied if the bottom consists of crests and valleys whose cross-sectional areas are the same and the bottom profile has a vertical plane of symmetry.

Let the ridge satisfy conditions (3.7) and be a local perturbation of the bottom, i.e., $h(x) = 0$ for $|x| > \lambda$. Then from (3.6) and (3.3) we can estimate the amplitude of the solitary wave perturbation caused by the ridge

$$\delta \sim \alpha\varepsilon\lambda^2. \tag{3.8}$$

In the dimensions of the variables

$$\delta_1 \sim \frac{\varepsilon_1 a^2 b^2}{h_0^4}. \tag{3.9}$$

Here ε_1 is the ridge height, b is the ridge width, and a is the amplitude of the solitary wave.

Note. It follows from (2.6), (2.7) that for steady wave propagation with the velocity $c = 1 + O(\alpha)$

$$\partial^2\varphi / \partial x^2 = O(\alpha + \varepsilon).$$

If we assume that

$$\frac{\partial}{\partial x} \sim \sqrt{\alpha}, \qquad \varepsilon = \alpha, \tag{3.10}$$

then after excluding ζ and neglecting terms of $O(\alpha)$ we obtain the following equation for the potential $\varphi(x, y)$ at the free surface:

$$\frac{\partial^2\varphi}{\partial X^2} - \left(\frac{c^2-1}{\alpha} + h(X)\right)\frac{\partial^2\varphi}{\partial y^2} + \frac{3c}{2}\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right)^2 + \frac{1}{3}\frac{\partial^4\varphi}{\partial y^4} = 0 \quad (X = \sqrt{\alpha}\,x). \tag{3.11}$$

We might think that this equation has a solution which decreases in all directions at infinity. But this is not the case, as is easily seen if we assume that the derivatives of order 1, 2, 3 of the function φ decrease at infinity no slower than $(X^2 + y^2)^{-1/2+\beta}$, where β is an arbitrarily small number. These conditions ensure that φ = const at ∞. Integration of (3.11) with respect to X for fixed y yields

$$\int_{-\infty}^{\infty}\left(\left(\frac{1-c^2}{\alpha} - h(X)\right)\frac{\partial\varphi}{\partial y} + \frac{3c}{2}\left(\frac{\partial\varphi}{\partial y}\right)^2 + \frac{1}{3}\frac{\partial^3\varphi}{\partial y^3}\right)dX = \text{const}. \tag{3.12}$$

It follows from the conditions at infinity that const = 0, after which it is not difficult to find by integrating (3.12) with respect to y that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial\varphi}{\partial y}\right)^2 dX\,dy=0 \quad \text{for} \quad \frac{\partial\varphi}{\partial y}\equiv 0. \tag{3.13}$$

4. Unsteady waves in a basin with a cylindrical bottom (linear theory). Let us examine under the assumptions of linear wave theory the unsteady propagation of waves in a basin with uneven bottom resulting from an initial disturbance. We assume the wave amplitude and its ratio to the wavelength to be small; this makes it possible to shift the conditions at the free surface to the plane of equilibrium of the resting fluid and neglect the nonlinear terms in the boundary conditions (1.3) and (1.4).

The basin depth is assumed to be arbitrary and independent of the variable y, i.e., the bottom is cylindrical $z = -1 + h(x)$, $h < 1$. The parameters α and ε equal unity. We seek the particular solutions of the problem which lead to an equation for the free surface of the form

$$\zeta = \psi(x)\, e^{i(\omega t - \nu y)}, \tag{4.1}$$

where ω and ν are parameters, and the functions $\psi(x) \to 0$ as $|x| \to \infty$. We thereby study solutions which represent a progressive wave propagating along the ridge and decaying rapidly in the direction perpendicular to the ridge.

Waves of this type, traveling along a sloping flat beach, were found by Stoker and have been studied by several authors [6].

Let the bottom be a horizontal plane with a local rise at $a < x < b$. Then it may be shown that solutions of the form (4.1) exist and have several interesting characteristics. We find that for fixed ν there is only a finite number of solutions of this form,

$$\psi = \psi_k(x, \nu),\ \omega = \omega_k(\nu),\ k = 1, 2, \ldots, n. \tag{4.2}$$

The functions $\psi_k(x, \nu)$ and the numbers $\omega_k(\nu)$ naturally depend on the parameter ν and are defined for $\alpha_k < |\nu| < \beta_k$, where the functions themselves, their number, and the interval of variation of ν are completely defined by the form of the bottom and are independent of the initial conditions [7]. The concrete computation of the functions ψ_k and ω_k and the determination of their number is possible only with the aid of computers. However, it is possible to make a qualitative analysis; for example, we can establish that for each

$$\psi_k(x, \nu) = O\left(e^{-p(\nu)|x|}\right),\ p(\nu) > 0 \ \text{for} \ |x| \to \infty. \tag{4.3}$$

Solution (4.1) does not approach 0 as $|y| \to \infty$. But if we multiply (4.1) by the arbitrary function $a(\nu)$ and integrate with respect to ν we obtain a solution which approaches 0 as $R = (x^2 + y^2)^{1/2} \to \infty$, and the rate of approach to infinity can be arbitrarily fast with suitable choice of the function $a(\nu)$. This wave solution is a cap-cloud type of wave which travels with constant mean velocity along the ridge, gradually spreading out along the ridge and reducing in amplitude. The energy of such a wave is localized in a band parallel to the ridge. Waves of this type also develop from an arbitrary initial disturbance.

Assume for simplicity that $\Phi(x, y, z, 0) = 0$. Then

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3 + \ldots + \zeta_n + \zeta_*, \tag{4.4}$$

where

$$\zeta_k = {}^1\!/_2\,\pi \int_{\alpha_k<|\nu|<\beta_k} e^{-i\nu y}\,\psi_k(x, \nu)\cos\omega_k(\nu)\,t\,a_k(\nu)\,d\nu, \tag{4.5}$$

$$a_k = \int e^{i\nu y}\,\psi_k(x, \nu)\,\zeta(x, y, 0)\,dx\,dy. \tag{4.6}$$

The sense of equality (4.4) is that the total motion energy

$$E(\zeta) = E(\zeta_1) + E(\zeta_2) + \ldots + E(\zeta_n) + E(\zeta_*). \tag{4.7}$$

Consequently the solution can be represented in the form of the sum of terms of a definite type. The terms ζ_k describe a group of waves traveling along the ridge. We can identify a special class of initial disturbances of the free surface when $\zeta_* = 0$. This class is fairly broad. For certain particular cases of initial conditions we may find that only the ζ_k terms are present in the solution, and the remaining terms of the series (4.4) vanish. Solution (4.4) is analogous to the expansion of an arbitrary motion of a linear oscillatory system into the sum of the characteristic oscillations.

A ledge on the bottom of the basin along a vertical beach plays exactly the same role as does the underwater ridge. What has been said above is also valid in this case.

5. Asymptotic behavior of unsteady waves above an underwater ridge. The study of the asymptotic behavior of the solutions through a long time interval is made using the stationary-phase method. For fixed x integral (4.5), which gives the solution ζ_k, differs from the solution for plane waves above a horizontal bottom only in the form of the function $\omega_k(\nu)$. The decay of ζ_k as $t \to \infty$, $y/t = \text{const}$, takes place because $d^2\omega_k/d\nu^2 \not\equiv 0$. The initial disturbance of the free surface breaks down into an ever increasing number of waves (crests and valleys), and the length of each wave (along the y-axis) increases. In this case the wave amplitude ζ_k decreases as $t^{-1/2}$ if $d^2\omega_k/d\nu^2 \neq 0$. However, if at some point $\nu_k \in (\alpha_k, \beta_k)$ the function $d^2\omega_k/d\nu^2$ has a zero of order m, then the vicinity of the point ν_k in integral (4.5) yields a group of relatively slowly disintegrating waves of amplitude $\sim t^{-1/(m+2)}$, traveling with the average velocity $d\omega_k(\nu_k)/d\nu$ along the y-axis. Study of the zeros of the function $d^2\omega_k/d\nu^2$ as a function of the ridge form shows that

$$\zeta_k \sim t^{-\gamma} \sim |y|^{-\gamma}, \tag{5.1}$$

where γ takes one of the values $1/2, 1/3, 1/4$ and is determined only by the shape of the bottom. Thus $\gamma = 1/4$ if the bottom shape satisfies a certain equality.

In order to analyze this phenomenon we use the approximate equations (2.6) and (2.7), which after linearization and simplification under assumption (3.10) yield the equation

$$\frac{\partial^2\zeta}{\partial t^2} = \alpha\frac{\partial^2\zeta}{\partial X^2} + (1-\alpha h(X))\frac{\partial^2\zeta}{\partial y^2} + \frac{\alpha}{3}\frac{\partial^4\zeta}{\partial y^4}, \quad X = \sqrt{\alpha}x. \tag{5.2}$$

This equation differs from the equation of linear shallow water theory in the last term. If we substitute into (5.2) a function of the form (4.1) we obtain the equation

$$\alpha\frac{d^2\psi}{dX^2} + \left(-\nu^2 + \omega^2 + \alpha\left(\nu^2 h(X) + \frac{\nu^4}{3}\right)\right)\psi(X) = 0. \tag{5.3}$$

Let

$$h(X) = \begin{cases} q, & |X| \leqslant 1 \\ 0, & |X| > 1 \end{cases}.$$

By virtue of the symmetry of the function h(X) the eigenfunctions of (5.3) can be only even and odd. Therefore it is sufficient to examine separately the even solutions

$$\psi = \begin{cases} a\cos\mu X & (0 < X < 1), \\ b\exp(-\sqrt{\nu^{*2} - \mu^2})(X-1) & (X > 1), \end{cases} \tag{5.4}$$

and the odd solutions, taking the sine in place of the cosine in the interval (−1, 1). Here

$$\mu^2 = \frac{(-\nu^2 + \omega^2)}{\alpha} + \nu^2 q + \frac{1}{3}\nu^4, \quad q > 0. \tag{5.5}$$

Substituting (5.4) into the matching conditions at the point X = 1 (continuity of ψ and $(1 - \alpha h)d\psi/dX$, or with the adopted accuracy $d\psi/dX$) we obtain

$$\mu_k = 1/2\,\pi\,(k-1) + \eta_k \qquad (0 \leqslant \eta_k < 1/2\pi). \tag{5.6}$$

Here η_k is defined by the implicit equation

$$1/2\pi\,(k-1) + \eta_k = \nu^*\cos\eta_k, \quad 1/2\pi\,(k-1) \leqslant \nu^* < \infty. \tag{5.7}$$

On the other hand, from definition (5.5) there follows to within $O(\alpha^2)$

$$\frac{d^2\omega_k}{d\nu^2} = \frac{\alpha}{\sqrt{q}}\left(\frac{q^2}{2}\frac{d^2}{d\nu^{*2}}\left(\frac{\mu_k^2(\nu^*)}{\nu^*}\right) - \nu^*\right). \tag{5.8}$$

It follows from (5.7) that in the vicinity of $\nu^* = 1/2\pi(k-1)$ the function

$$\frac{d^2}{d\nu^{*2}}\left(\frac{\mu_k^2}{\nu^*}\right) < 0$$

and is positive in the vicinity of $\nu^* = \infty$. Consequently, the right-hand side of (5.8) for sufficiently small q has no zeros; for sufficiently large q there are zeros. Numerical calculations show that for $q > q_k$ the function $d^2\omega_k/d\nu^2$ has two first-order zeros which merge for $q = q_k$, forming a second-order zero. We note that in the shallow water formulation the term $-\nu^*$ is not present in the bracket in (5.8); therefore there is a single first-order zero for all $q > 0$.

Thus, if the bottom is a broad step of small height then a very slowly decaying group of waves propagates along the ridge. Its rate of decay $|y|^{-\gamma}$ depends on the step area S as follows

$$\gamma = 1/2 \text{ for } 0 < 2q = S/h^2 < 33.6\,,$$
$$\gamma = 1/4 \text{ for } S/h^2_0 = 43.0, 33.6, 34.4, \ldots,$$
$$\gamma = 1/3 \text{ for other } S/h_0^2\,. \qquad (5.9)$$

Thus, the influence of an underwater ridge on wave propagation does not reduce to a simple increase of the amplitude but also determines in a significant way the very process of wave propagation, altering the nature of the wave decay along the ridge. Very slowly damped waves (5.1) can propagate along the ridge, while the asymptotic behavior of waves in a basin with smooth bottom has the form [8]

$$\zeta \sim (x^2 + y^2)^{-1/2}\,. \qquad (5.10)$$

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